

# Convergence of Infinite Series in Bosonic Second Quantization

Peter Otte  
Fakultät für Mathematik  
Ruhr-Universität Bochum  
Germany

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The functor of second quantization as well as quadratic creation and annihilation operators on the bosonic Fock space are defined through possibly infinite series. The domain of convergence is investigated by precise number operator estimates and the Banach-Steinhaus Theorem. Some fundamental properties of the limit operators are derived.

## 1 Introduction

The functor of second quantization is one of the so-called quadratic operators, which are used to quantize one-particle Hamiltonians, to construct representations of certain Lie algebras, and so on. There are three types of such operators, to wit

$$\begin{aligned} d\Gamma(B) &:= \sum_j a^\dagger(Be_j)a(\bar{e}_j), \\ \Delta(A) &:= \sum_j a(Ae_j)a(\bar{e}_j), \quad \Delta^+(C) := \sum_j a^\dagger(Ce_j)a^\dagger(\bar{e}_j). \end{aligned} \tag{1}$$

Here, we are working in a Fock representation of the canonical commutation relations (CCR) over a separable complex Hilbert space  $L$  with  $a(f)$  and  $a^\dagger(f)$  being the usual annihilation and creation operators on the Fock space  $\mathcal{F}$ .  $A, B, C$  are some linear operators on  $L$  and  $\{e_j\}$  is a complete orthonormal system (ONS) in  $L$ . Section 2 provides details and background.

When  $\dim L = \infty$  the sums in (1) are infinite series and, therefore, we need to look into convergence. Theorems 5.3, 5.7, 5.11 say that the series converge strongly on some

dense  $D_1 \subset \mathcal{F}$  which is what one can expect at best given that  $a(f)$  and  $a^\dagger(f)$  are unbounded. It turns out that  $d\Gamma(B)$  is well-defined for bounded  $B$  whereas  $\Delta(A)$  and  $\Delta^+(C)$  need Hilbert-Schmidt operators  $A, C$ . In some sense these conditions are necessary, see Propositions 5.4, 5.8, 5.12.

The proofs are based upon the Banach-Steinhaus Theorem 3.1, which actually deals with bounded operators. To make our operators bounded we define norms with the aid of the number operator  $N := d\Gamma(\mathbb{1})$ , which is studied thoroughly in Section 4. The difficult part then is to check the uniform boundedness condition in the Banach-Steinhaus Theorem. To this end, we derive precise number operator estimates for the partial sums of  $d\Gamma(B)$ ,  $\Delta(A)$ ,  $\Delta^+(C)$  in Lemmas 5.2, 5.6, 5.10 being true for the limit operators as well. The bounds for  $\Delta(A)$  and  $\Delta^+(C)$  improve those previously derived in [1, (2.1), (2.2)], [8, (2.3), (2.4)], [9, Lemma 3.6], [7, (70)].

All results can be transferred to fermionic operators, i.e. operators satisfying the canonical anticommutation relations instead. In that case, all number operator estimates can be strengthened provided the operators  $A, B, C$  belong to some von Neumann-Schatten class, see [11].

## 2 The CCR and Second Quantization

We will be working against the background of bosonic Fock space theory laid down in axioms (2) through (7) below (see also [12], [2], [10]).

Let  $L$  be a complex Hilbert space. For our purposes it is reasonable to require  $L$  to be separable with  $\dim L = \infty$ , without the latter some questions becoming trivial. Furthermore, there is a conjugation  $J : L \rightarrow L$ ,  $f \mapsto \bar{f}$ , compatible with the scalar product. By  $B_\infty(L)$  and  $B_p(L)$ ,  $p \geq 1$ , we denote the bounded operators and the von Neumann-Schatten classes on  $L$ , respectively. Finally, for  $A \in B_\infty(L)$  we define the conjugate  $\bar{A} := JAJ$  and the transpose  $A^T := \bar{A}^*$ .

Let  $\mathcal{F}$  be another complex Hilbert space. We take  $L$  as index space for operators in  $\mathcal{F}$ , i.e. an operator-valued functional is a map  $f \mapsto c(f)$  where  $c(f)$  is an operator in  $\mathcal{F}$  depending linearly on  $f$ . The CCR are concerned with two such functionals  $a, a^\dagger$ . We assume there is a common dense domain of definition  $D \subset \mathcal{F}$  with

$$a(f)D \subset D, \quad a^\dagger(f)D \subset D \quad \text{for all } f \in L. \quad (2)$$

These operators are said to give a representation of the CCR if for all  $f, g \in L$  on  $D$

$$[a(f), a(g)] = 0 = [a^\dagger(f), a^\dagger(g)], \quad (3)$$

$$[a(f), a^\dagger(g)] = (\bar{f}, g)\mathbb{1} \quad (4)$$

where the square brackets denote the commutator. We further require the unitarity condition, roughly saying  $a(f)^* = a^\dagger(\bar{f})$ . More precisely,

$$(a(f)\Phi, \Psi) = (\Phi, a^\dagger(\bar{f})\Psi) \text{ for all } \Phi, \Psi \in D. \quad (5)$$

A Fock representation has a vacuum  $\Omega \in \mathcal{F}$ ,  $\|\Omega\| = 1$ , that is annihilated by the  $a(f)$ 's

$$a(f)\Omega = 0 \text{ for all } f \in L \quad (6)$$

and cyclic for the  $a^\dagger(f)$ 's, i.e.

$$\mathcal{F} = \bar{\mathcal{F}}_0, \quad \mathcal{F}_0 := \text{span}\{a^\dagger(f_{j_n}) \cdots a^\dagger(f_{j_1})\Omega \mid n \in \mathbb{N}_0\}. \quad (7)$$

Consequently,  $a(f)$  and  $a^\dagger(f)$  are called annihilation and creation operator, respectively. Since most of the explicit calculations are carried out on  $\mathcal{F}_0$  we will use the notation

$$a^\dagger(f_n) \cdots \widehat{a^\dagger(f_j)} \cdots a^\dagger(f_1) := a^\dagger(f_n) \cdots a^\dagger(f_{j+1})a^\dagger(f_{j-1}) \cdots a^\dagger(f_1)$$

to indicate that a factor is missing. The structure of a Fock representation is fairly detailed determined by the axioms.

**Theorem 2.1.** (a) *The  $n$ -particle spaces  $\mathcal{F}^{(n)}$*

$$\mathcal{F}^{(n)} := \bar{\mathcal{F}}_0^{(n)}, \quad \mathcal{F}_0^{(n)} := \text{span}\{a^\dagger(f_n) \cdots a^\dagger(f_1)\Omega\}, \quad n \geq 0, \quad (8)$$

*are orthogonal to each other. The subspace of finite particle numbers*

$$\mathcal{F}_{fin} := \text{span}\{\Phi \mid \Phi \in \mathcal{F}^{(n)}, \quad n \in \mathbb{N}_0\} \quad (9)$$

*is dense in the Fock space,  $\mathcal{F} = \bar{\mathcal{F}}_{fin}$ , i.e.  $\mathcal{F}$  is the orthogonal sum of the  $n$ -particle spaces in the Hilbert space sense*

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}, \quad \Phi = \Phi^{(0)} + \Phi^{(1)} + \Phi^{(2)} + \cdots, \quad \Phi^{(n)} \in \mathcal{F}^{(n)}.$$

(b)  $\mathcal{F}^{(n)}$  and the symmetric tensor product  $L^{\otimes n}$  are isomorphic Hilbert spaces with the scalar products related via

$$(a^\dagger(f_n) \cdots a^\dagger(f_1)\Omega, a^\dagger(g_n) \cdots a^\dagger(g_1)\Omega) = \sum_{\sigma \in S_n} \prod_{j=1}^n (f_j, g_{\sigma(j)}) \quad (10)$$

where  $S_n$  are all permutations of  $\{1, \dots, n\}$ .

For the creation and annihilation operators it is obvious

$$a^\dagger(f) : \mathcal{F}_0^{(n)} \rightarrow \mathcal{F}_0^{(n+1)}, \quad a(f) : \mathcal{F}_0^{(n)} \rightarrow \mathcal{F}_0^{(n-1)}.$$

Furthermore, they are closable operators due to the unitarity condition (5) and  $D$  being dense. By the Wielandt-Wintner theorem [13], [14] they cannot be bounded on  $\mathcal{F}$ . However, they are bounded on the  $n$ -particle spaces.

**Theorem 2.2.** *The operators  $a(f)$  and  $a^\dagger(f)$  can be extended to  $\mathcal{F}^{(n)}$  and, thus, to  $\mathcal{F}_{fin}$ . The extended operators satisfy the CCR and*

$$\sup_{\substack{\Phi \in \mathcal{F}^{(n)} \\ \|\Phi\|=1}} \|a(f)\Phi\| = \sqrt{n}\|f\|, \quad \sup_{\substack{\Phi \in \mathcal{F}^{(n)} \\ \|\Phi\|=1}} \|a^\dagger(f)\Phi\| = \sqrt{n+1}\|f\|, \quad f \in L, \quad n \in \mathbb{N}_0. \quad (11)$$

Motivated by (11) we single out certain subspaces of  $\mathcal{F}$ .

**Theorem 2.3.** *Let  $\Phi \in \mathcal{F}$ ,  $\Phi = \Phi^{(0)} + \Phi^{(1)} + \dots$  with  $\Phi^{(n)} \in \mathcal{F}^{(n)}$ , and  $\alpha \geq 0$ . Then,*

$$D_\alpha := \{\Phi \in \mathcal{F} \mid \sum_{n=0}^{\infty} (n+1)^{2\alpha} \|\Phi^{(n)}\|^2 < \infty\}, \quad \|\Phi\|_\alpha^2 := \sum_{n=0}^{\infty} (n+1)^{2\alpha} \|\Phi^{(n)}\|^2$$

*is a Hilbert space with norm  $\|\cdot\|_\alpha$  the related properties having the prefix  $\alpha$ . The subspaces  $\mathcal{F}_0, \mathcal{F}_{fin} \subset D_\alpha$  are  $\alpha$ -dense and so is  $D_\beta \subset D_\alpha$  for  $\beta \geq \alpha$ .*

**Theorem 2.4.** *The operators  $a(f)$  and  $a^\dagger(f)$  can be extended to  $D_{1/2}$  with*

$$\|a(f)\Phi\| \leq \|f\| \|\Phi\|_{1/2}, \quad \|a^\dagger(f)\Phi\| \leq \|f\| \|\Phi\|_{1/2}.$$

*In particular, the maps  $f \mapsto a(f)\Phi$  and  $f \mapsto a^\dagger(f)\Phi$  are continuous for fixed  $\Phi \in D_{1/2}$ . Furthermore,  $a(f)$  and  $a^\dagger(f)$  satisfy the CCR on  $D_1$ .*

### 3 Prerequisites from Operator Theory

We collect the necessary prerequisites starting with the Banach-Steinhaus Theorem (BST).

**Theorem 3.1** (Banach-Steinhaus). *Let  $X$  and  $Y$  be Banach spaces and  $(F_n)$  be a sequence of bounded linear operators  $F_n : X \rightarrow Y$ . Then,  $(F_n)$  converges strongly on  $X$  if and only if*

1.  *$(F_n)$  converges strongly on a dense subset  $U \subset X$ .*
2. *The  $F_n$  are uniformly bounded,  $\|F_n\| \leq C$  for all  $n \in \mathbb{N}$  and some  $C$ .*

*Furthermore,  $F\varphi := \lim_{n \rightarrow \infty} F_n\varphi$  is bounded with  $\|F\| \leq C$ .*

To check the boundedness required by the BST we need some operator inequalities, which are to be understood in the sense of quadratic forms. At first, we generalize Cauchy-Schwarz's inequality.

**Lemma 3.2.** *Let  $\mathcal{H}$  be a Pre-Hilbert space and  $a_j, b_j : \mathcal{H} \rightarrow \mathcal{H}$  be linear operators with adjoints  $a_j^*, b_j^* : \mathcal{H} \rightarrow \mathcal{H}$ . Then,*

$$\pm \sum_{j,k=1}^M a_j^* b_k^* b_j a_k \leq \sum_{j,k=1}^M a_j^* b_k^* b_k a_j.$$

*Proof.* The inequality follows from

$$0 \leq \sum_{j,k=1}^M (\sigma b_k a_j - b_j a_k)^* (\sigma b_k a_j - b_j a_k) = 2 \sum_{j,k=1}^M (a_j^* b_k^* b_k a_j - \sigma a_j^* b_k^* b_j a_k)$$

where  $\sigma \in \{\pm 1\}$ . □

We reformulate some linear algebra within our framework.

**Lemma 3.3.** *Let  $a(f)$  and  $a^\dagger(f)$  be operator-valued functionals satisfying (2) and (5). Let  $A, B \in B_\infty(L)$  and  $\{e_j\}$ ,  $j = 1, \dots, M$ , be an ONS in  $L$ . Then,*

$$\begin{aligned} 0 &\leq \sum_{j=1}^M a^\dagger(BAe_j)a(\bar{B}\bar{A}\bar{e}_j) = \\ &= \sum_{j,k=1}^M (A^*e'_j, A^*e'_k)a^\dagger(Be'_j)a(\bar{B}\bar{e}'_k) \leq \|A\| \sum_{j=1}^M a^\dagger(Be'_j)a(\bar{B}\bar{e}'_j) \end{aligned} \quad (12)$$

where  $\{e'_j\}$  is any ONS such that  $\text{span}\{Ae_1, \dots, Ae_M\} \subset \text{span}\{e'_1, \dots, e'_M\}$ .

*Proof.* The restriction  $A_M$  of  $A$  to  $\text{span}\{e_1, \dots, e_M\}$  is compact. Therefore,

$$A_M g_j = \mu_j f_j, \quad A_M^* f_j = \mu_j g_j$$

with singular vectors and values. Obviously,

$$\begin{aligned} \sum_{j=1}^M a^\dagger(BA_M g_j)a(\bar{B}\bar{A}_M \bar{g}_j) &= \sum_{j=1}^M \mu_j^2 a^\dagger(Bf_j)a(\bar{B}\bar{f}_j) \\ &= \sum_{j,k=1}^M (A_M^* f_j, A_M^* f_k)a^\dagger(Bf_j)a(\bar{B}\bar{f}_k). \end{aligned}$$

To conclude the proof recall the estimate  $0 \leq \mu_j \leq \|A\| = \|A^*\|$  and note that in the first and last sum each ONS can be replaced by one that spans the same subspace. □

The preceding Lemmas 3.2 and 3.3 are concerned with quadratic forms, which of course can be written as inequalities involving the norm instead. For unbounded operators, like creation and annihilation operators, the norm inequalities make sense on a larger domain, however, without being proven there. This can be overcome by a simple observation, which becomes essential when treating the partial sums below.

**Lemma 3.4.** *Let  $X, Y$  be normed spaces and  $F : X \rightarrow Y$  be a bounded linear operator. If  $\|F\varphi\| \leq c\|\varphi\|$  on a dense subset  $U \subset X$  then  $\|F\| \leq c$ .*

*Proof.* Approximation argument. □

## 4 The Number Operator

To define quadratic operators we look at the simplest such operator  $N := d\Gamma(\mathbb{1})$ , the particle number operator or number operator for short. In what follows,  $\{e_j\}$  is a complete ONS in  $L$ . We study the partial sums

$$N_M := \sum_{j=1}^M a^\dagger(e_j)a(\bar{e}_j)$$

and work our way up to the maximal domain of convergence.

**Lemma 4.1.** *On  $\mathcal{F}_0$  the strong limit  $\lim_{M \rightarrow \infty} N_M =: N$  exists and reads*

$$N\Phi = n\Phi \text{ for } \Phi \in \mathcal{F}_0^{(n)}.$$

*Proof.* Take the limit  $M \rightarrow \infty$  in

$$N_M a^\dagger(f_n) \cdots a^\dagger(f_1) \Omega = \sum_{k=1}^n \sum_{j=1}^M (e_j, f_k) a^\dagger(e_j) a^\dagger(f_n) \cdots \widehat{a^\dagger(f_k)} \cdots a^\dagger(f_1) \Omega$$

and use that  $a^\dagger(f)\Phi$  is continuous in  $f$  by Theorem 2.2. □

The boundedness condition of the BST 3.1 ought to be intuitively clear.

**Lemma 4.2.** *The operator  $N_M$  satisfies*

$$\|N_M \Phi\| \leq \|\Phi\|_1 \text{ for } \Phi \in D_1.$$

*Proof.* It is enough to show the estimate on each  $\mathcal{F}^{(n)}$ . For  $n = 0$  it is trivial. Assume it is true for some  $n \geq 0$ . When  $\Phi \in \mathcal{F}^{(n+1)}$ ,

$$\begin{aligned}\|N_M \Phi\|^2 &= \sum_{j=1}^M (a(\bar{e}_j) \Phi, N_M a(\bar{e}_j) \Phi) + (\Phi, N_M \Phi) \\ &\leq \sum_{j=1}^M \|a(\bar{e}_j) \Phi\| \|N_M a(\bar{e}_j) \Phi\| + (\Phi, N_M \Phi)\end{aligned}$$

since  $a(f) \Phi \in \mathcal{F}^{(n)}$ . Note, trivially  $\mathcal{F}^{(n)} \subset D_{3/2}$ . Now,

$$\|N_M \Phi\|^2 \leq n \sum_{j=1}^M \|a(\bar{e}_j) \Phi\|^2 + (\Phi, N_M \Phi) = (n+1)(\Phi, N_M \Phi) \leq (n+1) \|\Phi\| \|N_M \Phi\|$$

completes the proof.  $\square$

**Theorem 4.3.** *The  $N_M$  converge strongly on  $D_1$  to a well-defined operator  $N$  with*

$$(N\Phi)^{(n)} = n\Phi^{(n)}. \quad (13)$$

*Proof.* The  $N_M$  converge strongly on the 1-dense subset  $\mathcal{F}_0 \subset D_1$ , Lemma 4.1, and their norms are bounded by Lemma 4.2. Hence, by the BST 3.1 the  $N_M$  converge on the whole  $D_1$  to a 1-bounded operator. The partial sums  $\tilde{N}_M$  formed with another ONS will converge as well to, say,  $\tilde{N}$ . By Lemma 4.1 on  $\mathcal{F}_0$  the limit looks the same whatever ONS we choose. Hence, the 1-bounded operators  $N$  and  $\tilde{N}$  coincide on the 1-dense subset  $\mathcal{F}_0$  and, thus, everywhere on  $D_1$ .  $\square$

We note some properties of  $N$ , in particular we relate the  $\alpha$ -norms to  $N$ .

**Theorem 4.4.**  *$N^\alpha$  is non-negative and self-adjoint on  $D_\alpha$  for  $\alpha \geq 0$ . Furthermore,  $\|\Phi\|_\alpha = \|(N + \mathbb{1})^\alpha \Phi\|$  for  $\Phi \in D_\alpha$ . On  $D_{3/2}$ ,*

$$[N, a(f)] = -a(f), \quad [N, a^\dagger(f)] = a^\dagger(f). \quad (14)$$

*Proof.* The spectral decomposition of  $N$  is given by (13) which implies  $N$  is non-negative and self-adjoint and also gives a representation for  $N^\alpha$ . The norm relation is obvious from Definition 2.3. On  $D_{3/2}$

$$\begin{aligned}\left[\sum_{j=1}^M a^\dagger(e_j) a(\bar{e}_j), a(f)\right] &= \sum_{j=1}^M [a^\dagger(e_j), a(f)] a(\bar{e}_j) = \\ &= -\sum_{j=1}^M (\bar{f}, e_j) a(\bar{e}_j) = -a\left(\sum_{j=1}^M (\bar{e}_j, f) \bar{e}_j\right).\end{aligned}$$

The leftmost term converges strongly on  $D_{3/2}$  to  $[N, a(f)]$  and the rightmost term to  $-a(f)$  since  $a(f)\Phi$  is continuous in  $f$ . The second relation follows in like manner.  $\square$

Because Lemma 4.1 needs a vacuum so does Theorem 4.3. However, there are representations other than the Fock representation that have a number operator when it is defined through strongly converging partial sums. There again, it can satisfy the commutation relations (14) only in a Fock representation. For a thorough discussion see [3], [4], [5], [6].

To treat general quadratic operators we need some technical estimates. To begin with, we rewrite Lemma 3.3 in a nearly obvious way.

**Lemma 4.5.** *Let  $A \in B_\infty(L)$ . Then, on  $D_1$*

$$\sum_{j=1}^M a^\dagger(Ae_j)a(\bar{A}\bar{e}_j) \leq \|A\|^2 N.$$

*Proof.* Apply (12) and bound the  $N_M$  by  $N$ .  $\square$

It is instructive to prove the next result with the aid of the commutators (14) instead of (13).

**Lemma 4.6.** *Let  $A \in B_\infty(L)$ ,  $\{e_j\}$  be an ONS in  $L$ , and  $f_j := Ae_j$ . Then, on  $D_2$*

$$\sum_{j=1}^M a^\dagger(f_j)(N + \mathbb{1})a(\bar{f}_j) \leq \|A\|^2 N^2 \text{ and } \sum_{j=1}^M a^\dagger(e_j)Na(\bar{e}_j) \leq N(N - \mathbb{1}).$$

*Proof.* With the commutator (14) and  $\gamma \in \mathbb{R} \setminus \{0\}$  on  $D_2$ ,

$$\begin{aligned} 2 \sum_{j=1}^M a^\dagger(f_j)Na(\bar{f}_j) &= - \left( \frac{1}{\gamma} \sum_{j=1}^M a^\dagger(f_j)a(\bar{f}_j) - \gamma N \right)^2 + \gamma^2 N^2 \\ &\quad + \frac{1}{\gamma^2} \sum_{j,k=1}^M a^\dagger(f_j)a^\dagger(f_k)a(\bar{f}_k)a(\bar{f}_j) \\ &\quad + \frac{1}{\gamma^2} \sum_{j,k=1}^M (f_j, f_k)a^\dagger(f_j)a(\bar{f}_k) - 2 \sum_{j=1}^M a^\dagger(f_j)a(\bar{f}_j). \end{aligned}$$

We drop the first term and estimate the quartic term and the first quadratic term via (12). To prove the first estimate choose  $\gamma = \|A\|$ , the case  $\|A\| = 0$  being trivial. To



prove the second note for every  $M' \geq M$

$$\sum_{j=1}^M a^\dagger(e_j) N a(\bar{e}_j) \leq \sum_{j=1}^{M'} a^\dagger(e_j) N a(\bar{e}_j) \leq N^2 - \sum_{j=1}^{M'} a^\dagger(e_j) a(\bar{e}_j)$$

by the first part. To conclude, let  $M' \rightarrow \infty$  and note the strong convergence on  $D_1$ .  $\square$

## 5 General Quadratic Operators

We are going to study the general quadratic operators from (1) by means of the respective partial sums

$$d\Gamma_M(B) := \sum_{j=1}^M a^\dagger(Be_j) a(\bar{e}_j),$$

$$\Delta_M(A) := \sum_{j=1}^M a(Ae_j) a(\bar{e}_j), \quad \Delta_M^+(C) := \sum_{j=1}^M a^\dagger(Ce_j) a^\dagger(\bar{e}_j).$$

Hereinafter,  $\{e_j\}$  always is a complete ONS in  $L$  and  $A, B, C \in B_\infty(L)$ . We will treat the three cases separately since the calculations differ in some points. However, the overall strategy is the same: The partial sums converge on  $\mathcal{F}_0$  and can be bounded with the aid of  $N$ . Then, the BST 3.1 extends convergence to  $D_1$ . We start with  $d\Gamma(B)$ .

**Lemma 5.1** (Convergence). *For  $B \in B_\infty$  the strong limit  $\lim_{M \rightarrow \infty} d\Gamma_M(B) =: d\Gamma(B)$  exists on  $\mathcal{F}_0$  with*

$$d\Gamma(B) a^\dagger(f_n) \cdots a^\dagger(f_1) \Omega = \sum_{j=1}^n a^\dagger(f_n) \cdots a^\dagger(Bf_k) \cdots a^\dagger(f_1) \Omega, \quad n \geq 1, \quad (15)$$

and  $d\Gamma(B) \Omega = 0$ .

*Proof.* Follows from

$$a^\dagger(Be_j) a(\bar{e}_j) a^\dagger(f_n) \cdots a^\dagger(f_1) \Omega = \sum_{k=1}^n (e_j, f_k) a^\dagger(Be_j) a^\dagger(f_n) \cdots \widehat{a^\dagger(f_k)} \cdots a^\dagger(f_1) \Omega \quad (16)$$

since  $f \mapsto a^\dagger(f) \Phi$  is continuous for fixed  $\Phi$  and  $B$  is bounded.  $\square$

**Lemma 5.2** (Boundedness). *Let  $B \in B_\infty(L)$ . Then, for  $\Phi \in D_1$*

$$\|d\Gamma_M(B) \Phi\| \leq \|B\| \|N \Phi\|.$$

*Proof.* After normal ordering we have on  $D_2$

$$\begin{aligned} d\Gamma_M(B)^* d\Gamma_M(B) &= \sum_{j,k=1}^M a^\dagger(e_j) a^\dagger(Be_k) a(\overline{Be_j}) a(\bar{e}_k) + \sum_{j,k=1}^M (Be_j, Be_k) a^\dagger(e_j) a(\bar{e}_k) \\ &\leq \|B\|^2 \sum_{j=1}^M a^\dagger(e_j) N a(\bar{e}_j) + \|B\|^2 \sum_{j=1}^M a^\dagger(e_j) a(\bar{e}_j) \end{aligned}$$

where we used the Cauchy-Schwarz inequality 3.2 and then Lemma 3.3. Now, Lemma 4.6 proves the estimate on  $D_2$ . To extend it we use the crude estimate on  $D_1$

$$\|d\Gamma_M(B)\Phi\| \leq \sum_{j=1}^M \|a^\dagger(Be_j) a(\bar{e}_j)\Phi\| \leq 2 \sum_{j=1}^M \|Be_j\| \|N\Phi\|.$$

Hence, Lemma 3.4 completes the proof.  $\square$

**Theorem 5.3.** *For  $B \in B_\infty(L)$  the partial sums  $d\Gamma_M(B)$  converge strongly on  $D_1$  to a well-defined operator  $d\Gamma(B)$  with*

$$\|d\Gamma(B)\Phi\| \leq \|B\| \|N\Phi\|, \quad \Phi \in D_1.$$

Furthermore,  $d\Gamma(B)^* = d\Gamma(B^*)$  on  $D_1$ .

*Proof.* By Lemma 5.1 the partial sums  $d\Gamma_M(B)$  converge on  $\mathcal{F}_0$ . Along with the bound from Lemma 5.2 the BST 3.1 implies strong convergence on  $D_1$  and the bound for  $d\Gamma(B)$ . If  $d\tilde{\Gamma}(B)$  is the limit obtained via another ONS it is an 1-bounded operator as well as  $d\Gamma(B)$ . By (15) they coincide on the 1-dense subset  $\mathcal{F}_0$  and, thus, everywhere on  $D_1$ . Likewise, using (15) and (10) one concludes that for  $\Phi, \Psi \in \mathcal{F}_0$

$$(\Phi, d\Gamma(B)\Psi) = (d\Gamma(B^*)\Phi, \Psi)$$

which extends to  $D_1$ . Thus,  $d\Gamma(B)^* = d\Gamma(B^*)$  on  $D_1$ .  $\square$

In a way,  $B$  being bounded is necessary when we want to have convergence on all of  $\mathcal{F}^{(1)}$ . If we drop this we can of course second quantize unbounded operators as well.

**Proposition 5.4.** *If  $B$  is defined on  $L_0 := \text{span}\{e_j \mid j \in \mathbb{N}\}$  and the corresponding  $d\Gamma_M(B)$  converge strongly on all of  $\mathcal{F}^{(1)}$  then  $B \in B_\infty(L_0)$ .*

*Proof.* Strong convergence on  $\mathcal{F}^{(1)}$  implies for all  $M$

$$\|d\Gamma_M(B)\Phi\| \leq c\|\Phi\|, \quad \Phi \in \mathcal{F}^{(1)},$$

by the BST 3.1 with some constant  $c$ . From this, (16) with  $n = 1$ , and (10) we get

$$\begin{aligned} c\|f\| &= c\|a^\dagger(f)\Omega\| \geq \|d\Gamma_M(B)a^\dagger(f)\Omega\| = \\ &= \|a^\dagger(B \sum_{j=1}^M (e_j, f)e_j)\Omega\| = \|B \sum_{j=1}^M (e_j, f)e_j\|. \end{aligned}$$

When  $f \in L_0$  choose  $M$  large enough and conclude that  $B$  is bounded on  $L_0$ .  $\square$

The  $\Delta_M(A)$  converge, initially, for a much larger class of operators  $A$ .

**Lemma 5.5** (Convergence). *Let  $A \in B_\infty(L)$ . The strong limit  $\lim_{M \rightarrow \infty} \Delta_M(A) =: \Delta(A)$  exists on  $\mathcal{F}_0$  and reads*

$$\Delta(A)a^\dagger(g_n) \cdots a^\dagger(g_1)\Omega = \sum_{\substack{k,l=1 \\ k \neq l}}^n (\bar{g}_l, Ag_k)a^\dagger(g_n) \cdots \widehat{a^\dagger(g_l)} \cdots \widehat{a^\dagger(g_k)} \cdots a^\dagger(g_1)\Omega. \quad (17)$$

*Proof.* Follows immediately from

$$\begin{aligned} a(Ae_j)a(\bar{e}_j)a^\dagger(g_n) \cdots a^\dagger(g_1)\Omega \\ = \sum_{\substack{k,l=1 \\ l \neq k}}^n (e_j, g_k)(\overline{Ae_j}, g_l)a^\dagger(g_n) \cdots \widehat{a^\dagger(g_l)} \cdots \widehat{a^\dagger(g_k)} \cdots a^\dagger(g_1)\Omega, \end{aligned}$$

the continuity of the scalar product, and the boundedness of  $A$ .  $\square$

**Lemma 5.6** (Boundedness). *Let  $A \in B_2(L)$ . Then on  $D_1$ ,*

$$\|\Delta_M(A)\Phi\|^2 \leq \|A\|^2 \|N\Phi\|^2 + (\|A\|_2^2 - \|A\|^2) \|N^{1/2}\Phi\|^2.$$

*Proof.* Let us write  $f_j := Ae_j$  for short. On  $D_2$ ,

$$\Delta_M(A)^* \Delta_M(A) = \sum_{j,k=1}^M a^\dagger(e_j)a(f_k)a^\dagger(\bar{f}_j)a(\bar{e}_k) - \sum_{j,k=1}^M (\bar{f}_k, \bar{f}_j)a^\dagger(e_j)a(\bar{e}_k).$$

The rightmost sum is positive by Lemma 3.3 and may be dropped. By Cauchy-Schwarz 3.2 and Lemmas 3.3 and 4.6

$$\begin{aligned} \Delta_M(A)^* \Delta_M(A) &\leq \sum_{j,k=1}^M a^\dagger(e_j)a^\dagger(\bar{f}_k)a(f_k)a(\bar{e}_j) + \sum_{j,k=1}^M (\bar{f}_k, \bar{f}_k)a^\dagger(e_j)a(\bar{e}_j) \\ &\leq \|A\|^2 \sum_{j=1}^M a^\dagger(e_j)Na(\bar{e}_j) + \|A\|_2^2 N \\ &\leq \|A\|^2 N(N-1) + \|A\|_2^2 N. \end{aligned}$$

This proves the estimate on  $D_2$ . On  $D_1$  we have

$$\|\Delta_M(A)\Phi\| \leq \sum_{j=1}^M \|Ae_j\| \|N\Phi\|$$

which allows us by dint of Lemma 3.4 to extend the estimate to  $D_1$ .  $\square$

The Hilbert-Schmidt condition, characteristic of the operator  $\Delta(A)$  as well as for  $\Delta^+(C)$  below, originates from this Lemma 5.6.

**Theorem 5.7.** *Let  $A \in B_2(L)$ . Then on  $D_1$  the partial sums  $\Delta_M(A)$  converge strongly to a well-defined operator  $\Delta(A)$  with*

$$\|\Delta(A)\Phi\|^2 \leq \|A\|^2 \|N\Phi\|^2 + (\|A\|_2^2 - \|A\|^2) \|N^{1/2}\Phi\|^2, \quad \Phi \in D_1.$$

Furthermore,  $\Delta(A^T) = \Delta(A)$ .

*Proof.* By Lemma 5.5 the  $\Delta_M(A)$  converge strongly on  $\mathcal{F}_0$ . Along with the bound from Lemma 5.6 the BST 3.1 implies strong convergence on  $D_1$  and the bound for  $\Delta(A)$ .  $\Delta(A)$  being independent of the ONS follows analogously to Theorem 5.3 via (17). Finally,

$$(\bar{g}_l, Ag_k) = (\bar{g}_k, A^T g_l)$$

in (17) implies that the 1-bounded operators  $\Delta(A)$  and  $\Delta(A^T)$  coincide on the 1-dense subspace  $\mathcal{F}_0 \subset D_1$  and, thus, on all of  $D_1$ .  $\square$

Even though we could define  $\Delta(A)$  on  $\mathcal{F}_0$  for bounded  $A$  the Hilbert-Schmidt condition in Theorem 5.7 happens to be necessary.

**Proposition 5.8.** *Let  $A^T = A$ . If  $\Delta_M(A)$  converges on  $\mathcal{F}^{(2)}$  then  $A \in B_2(L)$ .*

*Proof.* Standard calculations give the formula

$$\Delta_M(A)\Delta_M(A)^*\Omega = \left( \sum_{j,k=1}^M |(Ae_j, \bar{e}_k)|^2 + \sum_{k=1}^M \|Ae_k\|^2 \right) \Omega =: \omega_M \Omega. \quad (18)$$

Taking both the norm and the scalar product with  $\Omega$  yields

$$\|\Delta_M(A)\Delta_M(A)^*\Omega\| = \omega_M = \|\Delta_M(A)^*\Omega\|^2.$$

Let  $A \neq 0$  the case  $A = 0$  being trivial. Then  $\omega_M \neq 0$  for large  $M$ . Therefore, we can define unit vectors

$$\Phi_M := \frac{1}{\|\Delta_M(A)^*\Omega\|} \Delta_M(A)^*\Omega \in \mathcal{F}^{(2)}$$

and obtain

$$\|\Delta_M(A)\Phi_M\| = \omega_M^{\frac{1}{2}} \geq \left( \sum_{j=1}^M \|Ae_j\|^2 \right)^{\frac{1}{2}}.$$

When  $A \notin B_2(L)$  the right side will become infinite and so will the norm of  $\Delta_M(A)|_{\mathcal{F}^{(2)}}$ . By the BST 3.1 the  $\Delta_M(A)$  cannot converge on the entire Hilbert space  $\mathcal{F}^{(2)}$ .  $\square$

Unfortunately, one cannot deduce strong convergence of  $\Delta_M^+(C)$  from that of their adjoints  $\Delta_M(A)$ .

**Lemma 5.9** (Convergence). *For  $C \in B_2(L)$  the strong limit  $\lim_{M \rightarrow \infty} \Delta_M^+(C) =: \Delta^+(C)$  exists on  $\mathcal{F}_0$ .*

*Proof.*  $\Delta_M^+(C)$  and  $a^\dagger(f)$  commute. Since  $\Delta_M^+(C)\Omega \in \mathcal{F}^{(2)}$  and  $a^\dagger(f)|_{\mathcal{F}^{(n)}}$  is bounded it is enough to show convergence on  $\Omega$ . As in (18),

$$\left\| \sum_{j=M_1}^{M_2} a^\dagger(Ce_j) a^\dagger(\bar{e}_j) \Omega \right\|^2 = \sum_{j=M_1}^{M_2} \|Ce_j\|^2 + \sum_{j,k=M_1}^{M_2} (\bar{e}_j, Ce_k)(Ce_j, \bar{e}_k). \quad (19)$$

Since  $C \in B_2(L)$  the right side is a Cauchy sequence and so is  $\Delta_M^+(C)\Omega$ .  $\square$

**Lemma 5.10** (Boundedness). *Let  $C \in B_2(L)$  and  $\Phi \in D_1$ . Then,*

$$\|\Delta_M^+(C)\Phi\|^2 \leq \|C\|^2 \|(N(N+2\mathbb{1}))^{1/2}\Phi\|^2 + \|C\|_2^2 \|(N+2\mathbb{1})^{1/2}\Phi\|^2.$$

*Proof.* Put  $f_j := Ce_j$ . On  $D_2$ ,

$$\Delta_M^+(C)^* \Delta_M^+(C) = \sum_{j,k=1}^M a(\bar{f}_j) a^\dagger(\bar{e}_k) a(e_j) a^\dagger(f_k) + \sum_{j=1}^M a^\dagger(f_j) a(\bar{f}_j) + \sum_{j=1}^M \|f_j\|^2 \mathbb{1}.$$

With the aid of the Cauchy-Schwarz inequality 3.2

$$\sum_{j,k=1}^M a(\bar{f}_j) a^\dagger(\bar{e}_k) a(e_j) a^\dagger(f_k) \leq \sum_{j=1}^M a(\bar{f}_j) N a^\dagger(f_j).$$

Normal ordering and Lemma 4.6 imply the statement for  $D_2$ . On  $D_1$ ,

$$\|\Delta_M^+(C)\Phi\| \leq \sum_{j=1}^M \|Ce_j\| \|(N+1)\Phi\|.$$

Then, Lemma 3.4 shows the statement for  $D_1$ .  $\square$

**Theorem 5.11.** *If  $C \in B_2(L)$  the partial sums converge strongly on  $D_1$  to a well-defined operator  $\Delta^+(C)$  with*

$$\|\Delta^+(C)\Phi\|^2 \leq \|C\|^2\|(N(N+2\mathbb{1}))^{1/2}\Phi\|^2 + \|C\|_2^2\|(N+2\mathbb{1})^{1/2}\Phi\|^2, \quad \Phi \in D_1.$$

*Furthermore,  $\Delta^+(C^T) = \Delta^+(C)$  and  $\Delta^+(C) = \Delta(C^*)^*$  on  $D_1$  when  $C^T = C$ .*

*Proof.* By Lemmas 5.9, 5.10  $\Delta_M^+(C)$  converge on  $\mathcal{F}_0$  and are bounded in such a way that the BST 3.1 applies.  $\Delta^+(C)$  being independent of the ONS follows analogously to Theorem 5.3. Let  $\tilde{C} := C - C^T = -\tilde{C}^T$ . With the aid of (19),

$$\|(\Delta_M^+(C) - \Delta_M^+(C^T))\Omega\|^2 = \sum_{j=1}^M \|\tilde{C}e_j\|^2 - \sum_{j,k=1}^M |(\bar{e}_j, \tilde{C}e_k)|^2 \rightarrow 0, \quad M \rightarrow \infty,$$

which shows  $\Delta^+(C)$  and  $\Delta^+(C^T)$  coincide on  $\mathcal{F}_0$  and, thus, on  $D_1$  because of their being 1-bounded. Finally, when  $C^T = C$

$$(a(C^*e_j)a(\bar{e}_j))^* = a^\dagger(e_j)a^\dagger(C^T\bar{e}_j) = a^\dagger(C\bar{e}_j)a^\dagger(e_j).$$

This implies the statement for the adjoint since  $\Delta_M^+(C)$  and  $\Delta_M(C^*)$  converge independently of the ONS.  $\square$

Unlike  $\Delta(A)$  the operator  $\Delta^+(C)$  needs  $C \in B_2(L)$  to exist even on the vacuum.

**Proposition 5.12.** *Let  $C^T = C$ . If  $\Delta_M^+(C)$  converges on  $\Omega$  then  $C \in B_2(L)$ .*

*Proof.* Follows from (19) where the right side would become infinite if  $C \notin B_2(L)$ .  $\square$

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